Chapter 6 First - Order Transient Circuits

6.2 General form of the response equations

if you can solve a first order differential equation that means you able to solve the first order transient circuits.

The general form of the first order diff. Eq. Is

$$\frac{dx(t)}{dt} + ax(t) = f(t) \qquad 6.1$$

The solution of Eq. 6.1 is called the particular integral solution : x_p

And when the diff. Eq. 6.1 is called the homogeneous equation, the form of this equation is

$$\frac{dx(t)}{dt} + ax(t) = 0 \qquad 6.2$$

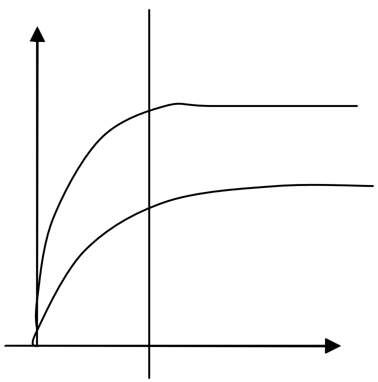
The solution of Eq. 6.2 is called the completed response or the natural response : x_c

Finally total of the solution is : X(t)

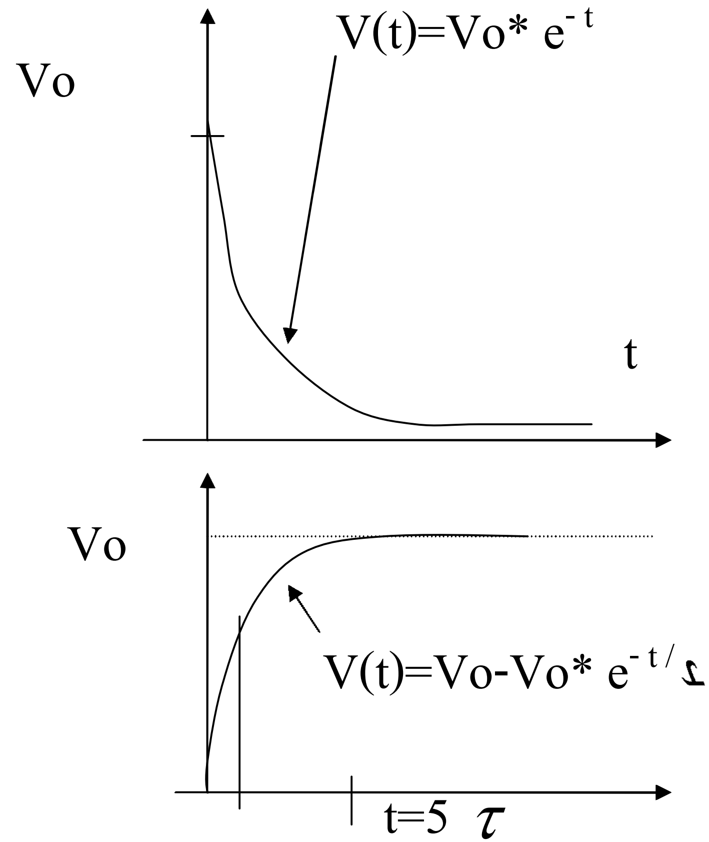
$$X(t) = x_p(t) + x_c(t) \qquad 6.3$$

$$\frac{dx(t)}{dt} + ax(t) = f(t)$$

$$X(t) = C1*e^{x(t)} + C2$$



$$V(t)=Vo\,(1- e^{-t/\tau})$$



At the present time we confine ourselves to the situation in which $f(t) = A$ (i.e., some constant). The general solution of the differential equation then consists of two parts that are obtained by solving the two equations

$$\frac{dx_p(t)}{dt} + ax_p(t) = A \tag{6.4}$$

$$\frac{dx_c(t)}{dt} + ax_c(t) = 0 \tag{6.5}$$

Since the right-hand side of Eq. (6.4) is a constant, it is reasonable to assume that the solution $x_p(t)$ must also be a constant. Therefore, we assume that

$$x_p(t) = K_1 \quad \frac{dx_p(t)}{dt} = 0 \tag{6.6}$$

Substituting this constant into Eq. (6.4) yields

$$K_1 = \frac{A}{a} \tag{6.7}$$

Examining Eq. (6.5), we note that

$$\frac{dx_c(t)/dt}{x_c(t)} = -a \tag{6.8}$$

This equation is equivalent to

$$\frac{d}{dt}\left[\ln x_c(t)\right] = -a$$

Hence,

$$\ln x_c(t) = -at + c$$

and therefore,

$$x_c(t) = K_2 e^{-at} \qquad 6.9$$

Therefore, a solution of Eq. (6.1) is

$$\begin{aligned} x(t) &= x_p(t) + x_c(t) \\ &= \frac{A}{a} + K_2 e^{-at} \end{aligned} \qquad 6.10$$

The constant K_2 can be found if the value of the independent variable $x(t)$ is known at one instant of time.

Equation (6.10) can be expressed in general in the form

$$x(t) = K_1 + K_2 e^{-t/\tau} \qquad 6.11$$

K1 is referred to as the *steady-state solution*: the value of the variable $x(t)$ as $t \to \infty$ when the second term becomes negligible. The constant τ is called the *time constant* of the circuit.

6.3 ANALYSIS TECHNIQUES

The Differential Equation Approach

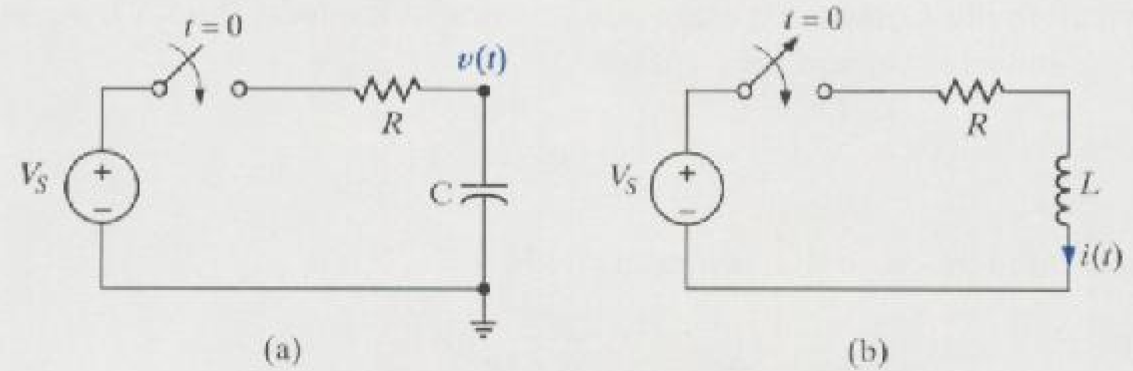


Figure 6.3 *RC* and *RL* circuits.

Consider the circuit shown in Fig. 6.3a. At time $t = 0$, the switch closes. The KCL equation that describes the capacitor voltage for time $t > 0$ is

$$C\frac{dv(t)}{dt} + \frac{v(t) - V_S}{R} = 0$$

or

$$\frac{dv(t)}{dt} + \frac{v(t)}{RC} = \frac{V_S}{RC}$$

From our previous development, we assume that the solution of this first-order differential equation is of the form

$$v(t) = K_1 + K_2 e^{-t/\tau}$$

Substituting this solution into the differential equation yields

$$-\frac{K_2}{\tau}e^{-t/\tau} + \frac{K_1}{RC} + \frac{K_2}{RC}e^{-t/\tau} = \frac{V_S}{RC}$$

Equating the constant and exponential terms, we obtain

$$K_1 = V_S$$
$$\tau = RC$$

Therefore,

$$v(t) = V_S + K_2 e^{-t/RC}$$

where V_S is the steady-state value and RC is the network's time constant. K_2 is determined by the initial condition of the capacitor. For example, if the capacitor is initially uncharged (that is, the voltage across the capacitor is zero at $t = 0$), then

$$0 = V_S + K_2$$

or

$$K_2 = -V_S$$

Hence, the complete solution for the voltage $v(t)$ is

$$v(t) = V_S - V_S e^{-t/RC}$$

The circuit in Fig. 6.3b can be examined in a similar manner. The KVL equation that describes the inductor current for $t > 0$ is

$$L\frac{di(t)}{dt} + Ri(t) = V_S$$

A development identical to that just used yields

$$i(t) = \frac{V_S}{R} + K_2 e^{-\left(\frac{R}{L}\right)t}$$

where V_S/R is the steady-state value and L/R is the circuit's time constant. If there is no initial current in the inductor, then at $t = 0$

$$0 = \frac{V_S}{R} + K_2$$

and

$$K_2 = \frac{-V_S}{R}$$

Hence,

$$i(t) = \frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{R}{L}t}$$

is the complete solution. Note that if we wish to calculate the voltage across the resistor, then

$$\begin{aligned} v_R(t) &= Ri(t) \\ &= V_S\left(1 - e^{-\frac{R}{L}t}\right) \end{aligned}$$

EXAMPLE 6.1

Consider the circuit shown in Fig. 6.4a. Assuming that the switch has been in position 1 for a long time, at time $t = 0$ the switch is moved to position 2. We wish to calculate the current $i(t)$ for $t > 0$.

SOLUTION At $t = 0-$ the capacitor is fully charged and conducts no current since the capacitor acts like an open circuit to dc. The initial voltage across the capacitor can be found using voltage division. As shown in Fig. 6.4b,

$$v_c(0-) = 12\left(\frac{3\text{k}}{6\text{k} + 3\text{k}}\right) = 4 \text{ V}$$

The network for $t > 0$ is shown in Fig. 6.4c. The KCL equation for the voltage across the capacitor is

$$\frac{v(t)}{R_1} + C\frac{dv(t)}{dt} + \frac{v(t)}{R_2} = 0$$

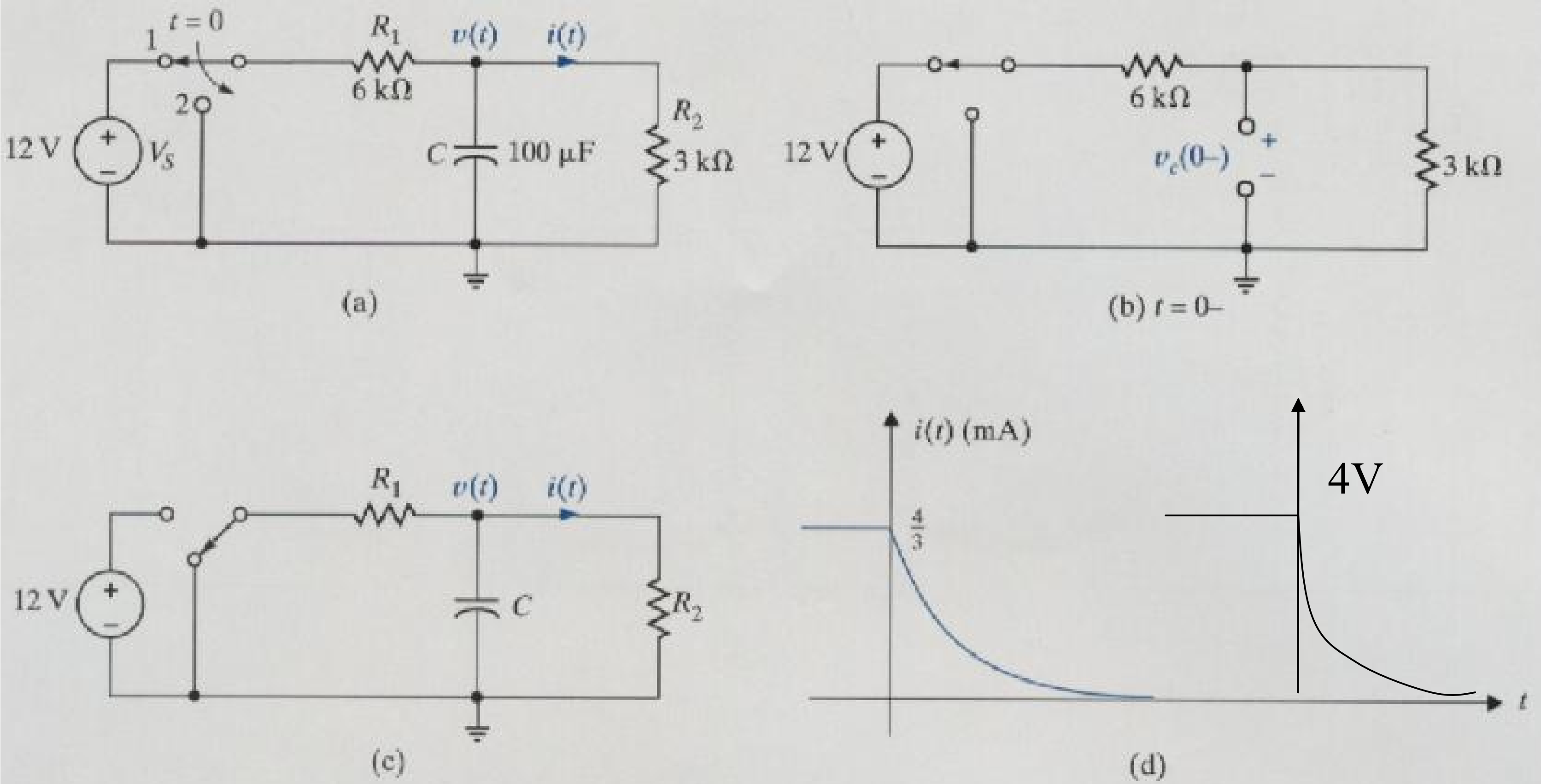


Figure 6.4 Analysis of *RC* circuits.

Using the component values, the equation becomes

$$\frac{dv(t)}{dt} + 5v(t) = 0$$

The form of the solution to this homogeneous equation is

$$v(t) = K_2 e^{-t/\tau}$$

If we substitute this solution into the differential equation, we find that $\tau = 0.2$ s. Thus,

$$v(t) = K_2 e^{-t/0.2} \text{ V}$$

Using the initial condition $v_c(0-) = v_c(0+) = 4$ V, we find that the complete solution is

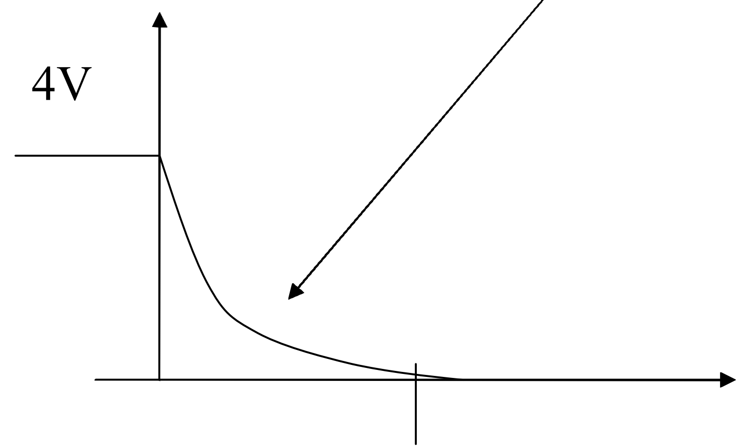
$$v(t) = 4e^{-t/0.2} \text{ V}$$

Then $i(t)$ is simply

$$i(t) = \frac{v(t)}{R_2}$$

or

$$i(t) = \frac{4}{3} e^{-t/0.2} \text{ mA}$$



$Vc = K1$ $t=0-$

$Vc = K2e^{-t/\tau}$ $t=0+$